

$$O_n = \{ g \in GL_n(\mathbb{R}) : {}^t g g = I_n \}$$

orthogonal sp

$$n \geq 1, n \in \mathbb{Z}$$

Recall: O_n is a sp

Propⁿ $O_n = \{ g \in GL_n(\mathbb{R}) : (gx) \cdot (gy) = x \cdot y \quad \forall x, y \in \mathbb{R}^n \}$.

Recall: If $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ then

$$x \cdot y = {}^t x \cdot y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

Pf of Propⁿ:

• Suppose $g \in O_n$. Then $(gx) \cdot (gy) = {}^t (gx)(gy)$
 $= {}^t x ({}^t g g) y = {}^t x I_n y = {}^t x y = x \cdot y \quad \forall x, y.$

Conversely, suppose $g \in GL_n(\mathbb{R})$ s.t. $(gx) \cdot (gy) = x \cdot y \quad \forall x, y \in \mathbb{R}^n$.

$$(gx) \cdot (gy) = {}^t x ({}^t g g) y = {}^t x y \quad \forall x, y \in \mathbb{R}^n.$$

$$\Leftrightarrow {}^t x ({}^t g g - I_n) y = 0 \quad \forall x, y \in \mathbb{R}^n.$$

Propⁿ follows from

Lemma: $A \in M_n(\mathbb{R})$ and ${}^t x A y = 0 \quad \forall x, y \in \mathbb{R}^n$. Then $A = 0$.

Pf: let $x = Ay$. Then

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$${}^t x Ay = {}^t (Ay) Ay = (Ay) \cdot (Ay) = 0 \quad \forall y \in \mathbb{R}^n.$$

$$\Rightarrow Ay = 0 \quad \forall y \in \mathbb{R}^n.$$

$$\Rightarrow A = 0.$$

$$SO_n = \{g \in O_n : \det(g) = 1\} \subseteq O_n, \text{ normal.}$$

Recall: $\det: O_n \rightarrow \{\pm 1\}$

$${}^t g g = I_n \text{ then } \det(g)^2 = 1 \Rightarrow \det(g) = \pm 1.$$

Set $r = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$ then $\begin{aligned} {}^t r r &= I_n \\ \det(r) &= -1. \end{aligned}$

Fact: If $x \in O_n$ then either

1) $x = g$ for some $g \in SO_n$

or 2) $x = gr$ for some $g \in SO_n$.

Pf: If $\det(x) = 1$ then in case 1.

$$\text{If } \det(x) = -1, \text{ set } g = xr. \text{ Then } \begin{aligned} \det(g) &= \det(x)\det(r) \\ &= (-1) \cdot (-1) \\ &= 1 \end{aligned}$$

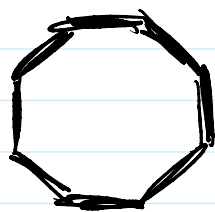
$$\Rightarrow x = gr^{-1} = gr.$$

Check this: r is a reflection in $\mathbb{R}^n$.

- We're going to:
- 1) Discuss symmetry in terms of G_n
 - 2) Relate SO_2 and SO_3 to rotations in $\mathbb{R}^2$ & $\mathbb{R}^3$

Isometry $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an isometry if it preserves distances between points, i.e.

$$\|f(u) - f(v)\| = \|u - v\| \quad \text{for all } u, v \in \mathbb{R}^n.$$



When we say that the octagon has lots of symmetry, we mean that there are nontrivial isometries of $\mathbb{R}^2$ that take the octagon to itself.

Examples of isometries

1) If $a \in \mathbb{R}^n$, then $t_a: \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined as $t_a(x) = x + a$ (i.e., translation by a) is an isometry.

2) If $\varphi \in G_n$, then φ is an isometry

$$\begin{aligned} \|\varphi(u) - \varphi(v)\| &= ((\varphi(u) - \varphi(v)) \cdot (\varphi(u) - \varphi(v)))^{1/2} \\ &= (\varphi(u-v) \cdot \varphi(u-v))^{1/2} \\ &= ((u-v) \cdot (u-v))^{1/2} \\ &= \|u-v\| \end{aligned}$$

3) If f_1 and f_2 are isometries then $f_1 \circ f_2: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an isometry. In particular, if $\varphi \in O_n$ then $t_a \circ \varphi$ is an isometry.

Thm: Every isometry f of $\mathbb{R}^n$ is of the form $t_a \circ \varphi$ for some $a \in \mathbb{R}^n$ and $\varphi \in O_n$.

Pf: Set $a = f(0)$, $\varphi = t_a^{-1} \circ f$, $t_a^{-1} = t_{-a}$ is the inverse of t_a .

$$\text{Then } \varphi(0) = t_a^{-1} f(0) = t_a^{-1}(a) = 0.$$

$$f = t_a \circ \varphi.$$

Require the following:

Thm: $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}^n$. TFAE.

a) φ isometry & $\varphi(0) = 0$

b) φ preserves dot products, i.e. $\varphi(u) \cdot \varphi(v) = u \cdot v \quad \forall u, v \in \mathbb{R}^n$

c) $\varphi \in O_n$.

Pf: (c) $\Rightarrow$ (a) $\checkmark$

Lemma: $x, y \in \mathbb{R}^n$. If

$$x \cdot x = x \cdot y = y \cdot y \quad \text{then } x = y.$$

Pf of Lemma $(x-y) \cdot (x-y) = x \cdot x - 2x \cdot y + y \cdot y$
 $= 0$

$$= 0$$

$$\Rightarrow x-y=0 \Rightarrow x=y$$

Will use lemma for (b) $\Rightarrow$ (c).

(a) $\Rightarrow$ (b) / φ isometry

$$\Rightarrow (\varphi(u) - \varphi(v)) \cdot (\varphi(u) - \varphi(v)) = (u-v) \cdot (u-v) \quad \forall u, v \in \mathbb{R}^n$$

Expanding:

$$\varphi(u) \cdot \varphi(u) - 2\varphi(u) \cdot \varphi(v) + \varphi(v) \cdot \varphi(v) = \underbrace{u \cdot u - 2u \cdot v + v \cdot v}_{\forall u, v \in \mathbb{R}^n}$$

Plugging in $v=0$ $\varphi(0)=0 \Rightarrow$

$$\text{LHS: } \varphi(u) \cdot \varphi(u) \Rightarrow \varphi(u) \cdot \varphi(u) = u \cdot u \quad \forall u \in \mathbb{R}^n$$

$$\text{RHS: } u \cdot u$$

$$-2\varphi(u) \cdot \varphi(v) = -2u \cdot v \quad \forall u, v \in \mathbb{R}^n$$

$$\Rightarrow \varphi(u) \cdot \varphi(v) = u \cdot v \quad \forall u, v \in \mathbb{R}^n$$

I.e. φ preserves dot products

Now (b) $\Rightarrow$ (c) : We're given φ preserves dot products, i.e. $\varphi(u) \cdot \varphi(v) = u \cdot v \quad \forall u, v \in \mathbb{R}^n$.

What we need to prove: φ is linear, i.e. $\varphi(u+v) = \varphi(u) + \varphi(v)$
and $\varphi(cu) = c\varphi(u)$

$$\forall u, v \in \mathbb{R}^n, c \in \mathbb{R}$$

$$\perp \quad \forall u, v \in K, \quad c \in K$$

Will apply lemma w/ $x = \varphi(u) + \varphi(v)$, $y = \varphi(u+v)$.

WT Check:

$$\begin{aligned} & (\varphi(u) + \varphi(v)) \cdot (\varphi(u) + \varphi(v)) \stackrel{?}{=} (\varphi(u) + \varphi(v)) \cdot \varphi(u+v) \stackrel{?}{=} \varphi(u+v) \cdot \varphi(u+v) \\ & \quad \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \qquad \qquad \parallel \left\| \begin{array}{l} \varphi \text{ preserves} \\ \text{dot product} \end{array} \right. \\ & \varphi(u) \cdot \varphi(u) + 2\varphi(u) \cdot \varphi(v) + \varphi(v) \cdot \varphi(v) \qquad \varphi(u) \cdot \varphi(u+v) + \varphi(v) \cdot \varphi(u+v) \\ & \quad \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \\ & u \cdot u + 2u \cdot v + v \cdot v = u \cdot (u+v) + v \cdot (u+v) \stackrel{\checkmark}{=} (u+v) \cdot (u+v) \end{aligned}$$

$$\Rightarrow \varphi(u+v) = \varphi(u) + \varphi(v)$$

Finally, must check $\varphi(cu) = c\varphi(u) \quad \forall c \in \mathbb{R}, u \in \mathbb{R}^n$.

Suffices to verify:

$\varphi(cu) \cdot \varphi(cu) \stackrel{?}{=} (c\varphi(u)) \cdot \varphi(cu) \stackrel{?}{=} (c\varphi(u)) \cdot (c\varphi(u))$

To apply lemma w/ $x = \varphi(cu)$, $y = c\varphi(u)$,

$(cu) \cdot (cu) \quad \checkmark \quad c(\varphi(u) \cdot \varphi(cu)) \quad \checkmark \quad c^2 v \cdot v$

$\quad \quad \quad \parallel \quad \quad \quad \parallel$

$\quad \quad \quad c(v \cdot cu) \quad \quad \quad v \cdot v$

$$\Rightarrow (C_b) \Rightarrow (C)$$

Just checked: φ preserves dot products $\implies \varphi$ is linear
+ φ preserves

Just checked: φ preserves dot product $\rightarrow \varphi$ is linear
+ φ preserves dot product

Must still verify: $\varphi \in GL_n(\mathbb{R})$

Suffices to check that $\ker(\varphi) = \{0\}$ as a linear operator.

Suppose $v \in \mathbb{R}^n$, $\varphi(v) = 0$. Well,

$$0 = \varphi(v) \cdot \varphi(v) \stackrel{\varphi \text{ pres. dot product}}{=} v \cdot v \Rightarrow v = 0. \quad \square$$